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Gauge Invariance in Quantum Electrodynamics

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Contrary to the conventional view point of quantization that breaks the gauge symmetry, a gauge invariant formulation of quantum electrodynamics is proposed. Instead of fixing the gauge, some frame is chosen to yield the locally invariant fields. We show that all the formulations, such as the Coulomb, the axial, and the Lorentz gauges, can be constructed and that the explicit LSZ mapping connecting Heisenberg operators to those of the asymptotic fields is possible. We also make some comments on gauge transformations in quantized field theory.

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Symmetry plays an important role in physics. Usually it must be kept as precise as possible, but as for the gauge symmetry in quantum field theories the scenario is completely different: we first *break the symmetry*, that is, fix the gauge then quantize. The situation is most easily seen by the recipe of Dirac [1]; start with the usual Lagrangian,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) - J^{\mu}(x)A_{\mu}(x), \tag{1}$$

where $J^{\mu}(x)$ is the matter current. Then we have the first-class constraints, $\Phi_1(\mathbf{x}) \equiv \Pi^0(\mathbf{x})$, and $\Phi_2(\mathbf{x}) \equiv \sum_{k=1}^{3}\left(\partial_k \Pi^k(\mathbf{x})\right) + J^0(\mathbf{x})$, where $\Pi^{\mu}(\mathbf{x})'$s are the canonical conjugate momenta. They form the the generator of gauge transformation,

$$A_{\mu}(x) \mapsto A_{\mu}(x) + \partial_{\mu}\chi(x), \tag{2}$$

such that

$$Q^{\chi}(x^0) \equiv \int d^3\mathbf{x}\Big(\dot{\chi}(x)\Phi_1(\mathbf{x}) - \chi(x)\Phi_2(\mathbf{x})\Big), \tag{3}$$

with $\left\{A_{\mu}(\mathbf{x}), Q^{\chi}(x^0)\right\}_{\mathrm{P}} = \partial_{\mu}\chi(x)$, where $\{A, B\}_{\mathrm{P}}$ is the Poisson bracket. A way to quantization can be seen by the introduction of a gauge condition which renders these first-class constraints into the second class ones obtain the Dirac bracket $\{A, B\}_{\mathrm{D}}$ and then by the corresponding rule: $\{A, B\}_{\mathrm{D}} \mapsto [A, B]/i\hbar$. Thus we obtain the canonical operator formalism of quantum electrodynamics.

The important fact is that the generator of gauge transformation, $\Phi_1(\mathbf{x})$ and $\Phi_2(\mathbf{x})$, had been exhausted to set up the Dirac bracket so that *there remains no freedom of gauge transformation at all in quantum electrodynamics.* In other words, quantization in various gauges is expressed by different commutation relations to yield different Hilbert spaces, so that each (quantized) theory should be regarded as *independent* of others [2]. Although this might be due to the situation that 'gauge symmetry is not a symmetry but a redundancy [3],' the fact that the S-matrix has been proved to be gauge invariant [4] enables us to assume that 'all physical quantities become gauge invariant at the end', which might be the spirit of gauge symmetry. Nowadays the trend of thinking that gauge variant quantities are not physical observables is widely spread out, owing to the issue of quark confinement in nonabelian gauge theories, and is especially emphasized in lattice gauge theory (which preserves the gauge symmetry at the sacrifice of the Lorentz invariance)[5]. However consider the canonical energy-momentum tensor,

$$\begin{aligned} T_{\mu\nu}(x) &\equiv \sum_a \frac{\partial \mathcal{L}}{\partial(\partial^{\mu}\phi^a(x))}\partial_{\nu}\phi^a(x) - g_{\mu\nu}\mathcal{L} \\ &\equiv T_{\mu\nu}\left(\partial_{\mu}\phi^a(x), \partial_{\mu}A_{\nu}(x); \Big(\partial_{\mu} - ie^a A_{\mu}(x)\Big)\phi^a(x), F_{\mu\nu}(x)\right), \end{aligned} \tag{4}$$

where e^a is the electric charge of the field $\phi^a(x)$. It is apparently gauge variant, so is the energy-momentum,

$$P_\mu = \int d^3\mathbf{x}\ T_{0\mu}(x), \tag{5}$$

as well as the Heisenberg equations of motion,

$$i\partial_\mu \phi^a(x) = \Big[\phi^a(x), P_\mu\Big]. \tag{6}$$

Therefore the energy-momentum is gauge variant. Hence it is unphysical! (It should be noted that (6) can be considered as a starting point of quantum theory [6].) Of course, we can build a gauge invariant energy-momentum tensor,

$$T^{\mathrm{G}}_{\mu\nu}(x) \equiv T_{\mu\nu}\Bigg(\Big(\partial_\mu - ie^a A_\mu(x)\Big)\phi^a(x), F_{\mu\nu}(x); \Big(\partial_\mu - ie^a A_\mu(x)\Big)\phi^a(x), F_{\mu\nu}(x)\Bigg) \tag{7}$$

by differentiating the Lagrangian on a curved manifold with respect to $g_{\mu\nu}(x)$ and putting $g_{\mu\nu}(x) \rightarrow\ g_{\mu\nu}$: diag $g_{\mu\nu} = (+,-,-,-)$) or with the aid of the method introduced by one of the authors [7]. The gauge covariant energy-momentum,

$$P^G_\mu \equiv \int d^3\mathbf{x}\ T^{\mathrm{G}}_{0\mu}(x), \tag{8}$$

should then imply the gauge covariant Heisenberg equation,

$$i\big(\partial_\mu - ie^a A_\mu(x)\big)\phi^a(x) = \Big[\phi^a(x), P^G_\mu\Big], \tag{9}$$

which however leads us, with the aid of the Jacobi identity, to

$$\Big[P^G_\mu, P^G_\nu\Big] \neq 0. \tag{10}$$

This shows that we cannot diagonalize the energy-momentum (8) simultaneously such that $P^G_\mu|p\rangle = p_\mu|p\rangle$. It is also obvious that we cannot use the perturbation approach to explore the states, $|p\rangle$'s, since the zeroth approximation setting $eA_\mu(x) = 0$ breaks the gauge invariance!

Motivated by these, we shall clarify in the following the meaning of the gauge field and its interaction. To this end let us recall the gauge invariant quantities in quantum electrodynamics: the minimal coupling term,

$$\bar{\psi}(x)i\gamma^\mu\big(\partial_\mu - ieA_\mu(x)\big)\psi(x), \tag{11}$$

and the field strength tensor $F_{\mu\nu}(x)$.

The gauge transformation is expressed as (2) together with

$$\begin{aligned} \psi(x) \mapsto & e^{ie\chi(x)}\psi(x), \\ \bar{\psi}(x) \mapsto & \bar{\psi}(x)e^{-ie\chi(x)}. \end{aligned} \tag{12}$$

In terms of the components, (2) reads as

$$\begin{aligned} A_0(x) \mapsto & A_0(x) + \dot{\chi}(x), \\ \mathbf{A}(x) \mapsto & \mathbf{A}(x) - \nabla\chi(x). \end{aligned} \tag{13}$$

Now we decompose the vector potential $\mathbf{A}(x)$ into

$$\mathbf{A}(x) = \mathbf{A}_{\mathrm{T}}(x) + \mathbf{A}_{\mathrm{L}}(x), \tag{14}$$

where $\mathbf{A}_{\mathrm{T}}(x)(\mathbf{A}_{\mathrm{L}}(x))$ denotes the transverse (longitudinal) component with respect to the derivative ∇; thus

$$\begin{aligned} \nabla \cdot \mathbf{A}_{\mathrm{T}}(x) =& 0, \\ \nabla \times \mathbf{A}_{\mathrm{L}}(x) =& 0. \end{aligned} \tag{15}$$

In view of (13) , we obtain the transformation rule:

$$\begin{aligned} \mathbf{A}_{\mathrm{T}}(x) \mapsto & \mathbf{A}_{\mathrm{T}}(x), \\ \mathbf{A}_{\mathrm{L}}(x) \mapsto & \mathbf{A}_{\mathrm{L}}(x) - \nabla\chi(x), \end{aligned} \tag{16}$$

that is, $\mathbf{A}_{\mathrm{T}}(x)$ is *gauge invariant.* In order to find other invariant quantities, let us go back to (11). First it should be noticed that

$$\begin{aligned} \psi^{\mathrm{c}}_{\mathrm{inv}}(x) \equiv & \exp\left[ie\int^{\mathbf{x}} d\mathbf{z}\cdot\mathbf{A}_{\mathrm{L}}(x^0,\mathbf{z})\right]\psi(x), \\ \bar{\psi}^{\mathrm{c}}_{\mathrm{inv}}(x) \equiv & \bar{\psi}(x)\exp\left[-ie\int^{\mathbf{x}} d\mathbf{z}\cdot\mathbf{A}_{\mathrm{L}}(x^0,\mathbf{z})\right], \end{aligned} \tag{17}$$

are gauge invariant under (12) and (16), path-independent owing to (15) (hence the beginning point of the integral can be arbitrary), and *in fact local* contrary to the Dirac's physical electron[8]. Then the minimal coupling term (11) becomes

$$\bar{\psi}^{\mathrm{c}}_{\mathrm{inv}}(x)i\Big[\gamma^0\big\{\partial_0 - ie\big(A_0(x) + \int^{\mathbf{x}} d\mathbf{z}\cdot\dot{\mathbf{A}}_{\mathrm{L}}(x^0,\mathbf{z})\big)\big\} - \gamma\cdot\big(\nabla + ie\mathbf{A}_{\mathrm{T}}(x)\big)\Big]\psi^{\mathrm{c}}_{\mathrm{inv}}(x), \tag{18}$$

yielding the gauge invariant potential,

$$\overset{\mathrm{c}}{A}_\mu(x) \equiv \big(\overset{\mathrm{c}}{A}_0(x), \overset{\mathrm{c}}{\mathbf{A}}(x)\big), \tag{19}$$

with $\overset{\mathrm{c}}{\mathbf{A}}(x) \equiv \mathbf{A}_{\mathrm{T}}(x)$, and

$$\overset{\mathrm{c}}{A}_0(x) \equiv A_0(x) + \int^{\mathbf{x}} d\mathbf{z} \cdot \dot{\mathbf{A}}_{\mathrm{L}}(x^0, \mathbf{z}). \tag{20}$$

Apparently

$$\nabla \cdot \overset{\mathrm{c}}{\mathbf{A}}(x) = 0. \tag{21}$$

In view of (21) this looks like the Coulomb gauge case but we *did not fix the gauge at all*, instead we have chosen the special frame which enables us to decompose the vector potential as in (14) and (15): indeed in a similar manner, take *some vector* $\mathbf{n}$; $\mathbf{n} \cdot \mathbf{n} = 1$. Then as in (14) and (15) we obtain

$$\begin{aligned} \mathbf{A}_\perp(x) &= \mathbf{A}(x) - \mathbf{A}_\parallel(x), \\ \mathbf{A}_\parallel(x) &\equiv \mathbf{n}\big(\mathbf{n} \cdot \mathbf{A}(x)\big), \end{aligned} \tag{22}$$

so that

$$\begin{aligned} \mathbf{n} \cdot \mathbf{A}_\perp(x) &= 0, \\ \mathbf{n} \times \mathbf{A}_\parallel(x) &= 0. \end{aligned} \tag{23}$$

The gauge transformation (13) becomes

$$\begin{aligned} \mathbf{A}_\perp(x) &\mapsto \mathbf{A}_\perp(x) - \nabla_\perp \chi(x), \\ \mathbf{A}_\parallel(x) &\mapsto \mathbf{A}_\parallel(x) - \nabla_\parallel \chi(x), \end{aligned} \tag{24}$$

with $\nabla_\perp \equiv \nabla - \nabla_\parallel$ and $\nabla_\parallel \equiv \mathbf{n}(\mathbf{n} \cdot \nabla)$, which, unlike the previous case, shows that both components are transformed. Invariant fermion fields are given by

$$\begin{aligned} \psi^{\mathrm{a}}_{\mathrm{inv}}(x) \equiv& \exp\Big[ie \int^{\mathbf{x}_\parallel} d\mathbf{z} \cdot \mathbf{A}_\parallel(\hat{x}, \mathbf{n} \cdot \mathbf{z})\Big] \psi(x), \\ \bar{\psi}^{\mathrm{a}}_{\mathrm{inv}}(x) \equiv& \bar{\psi}(x) \exp\Big[-ie \int^{\mathbf{x}_\parallel} d\mathbf{z} \cdot \mathbf{A}_\parallel(\hat{x}, \mathbf{n} \cdot \mathbf{z})\Big], \end{aligned} \tag{25}$$

where $\hat{x}$ denotes the rest of the components other than $\mathbf{n} \cdot \mathbf{z}$. They are again the local quantities. The invariant potential is again found by substituting (25) into (11), to be

$$\overset{\mathrm{a}}{A}_\mu(x) \equiv \big(\overset{\mathrm{a}}{A}_0(x), \overset{\mathrm{a}}{\mathbf{A}}(x)\big), \tag{26}$$

with

$$
\begin{aligned}
\overset{\mathrm{a}}{A}_0(x) &\equiv A_0(x) + \int^{\mathbf{x}_\parallel} d\mathbf{z}\cdot\dot{\mathbf{A}}_\parallel(\hat{x}, \mathbf{n}\cdot\mathbf{z}),\\
\overset{\mathrm{a}}{\mathbf{A}}(x) &\equiv \mathbf{A}_\perp(x) - \nabla_\perp \int^{\mathbf{x}_\parallel} d\mathbf{z}\cdot\mathbf{A}_\parallel(\hat{x}, \mathbf{n}\cdot\mathbf{z}),
\end{aligned}
\tag{27}
$$

obeying

$$
\mathbf{n}\cdot\overset{\mathrm{a}}{\mathbf{A}}(x) = 0. \tag{28}
$$

This is called as the *axial gauge*.

A few comments are in order: the traditional view point of fixing the gauge has been taken over to the new one of choosing a frame in which (three-dimensional) vector potential is divided into the parallel and the perpendicular components with respect to some vector (such as the ∇ or the unit vector $\mathbf{n}$) to form gauge invariant quantities. Hence the result would depend on the frame, then *to check the gauge invariance is nothing but to show that the result is covariant.* This is indeed the case as far as the perturbation theory is concerned; since the propagator in the axial gauge, for example, is given by

$$
D_{\mu\nu}(q) = \frac{-1}{q^2 + i\epsilon}\big(g_{\mu\nu} - \frac{\eta_\mu q_\nu - \eta_\nu q_\mu}{(\eta q)} - \frac{q_\mu q_\nu}{(\eta q)^2}\big) \tag{29}
$$

with $\eta^\mu \equiv (0, \mathbf{n})$, whose momentum-dependent numerators are *frame-dependent thus break the covariance and have been called as 'the gauge terms'*.

The recipe can immediately be applied to the covariant case: the decomposition is read as

$$
A_\mu(x) = A^{\mathrm{T}}_\mu(x) + A^{\mathrm{L}}_\mu(x), \tag{30}
$$

with

$$
\begin{aligned}
\partial^\mu A^{\mathrm{T}}_\mu(x) &= 0,\\
\partial_\mu A^{\mathrm{L}}_\nu(x) - \partial_\nu A^{\mathrm{L}}_\mu(x) &= 0,
\end{aligned}
\tag{31}
$$

where we have employed the superscript notation of T and L in order to distinguish this from the Coulomb case. In view of (2) and (30), the invariant vector potential in this case is $A^{\mathrm{T}}_\mu(x) : A^{\mathrm{T}}_\mu(x) \mapsto A^{\mathrm{T}}_\mu(x)$, but $A^{\mathrm{L}}_\mu(x) \mapsto A^{\mathrm{L}}_\mu(x) + \partial_\mu\chi(x)$. Now write

$$
\mathcal{A}_\mu(x) \equiv A^{\mathrm{T}}_\mu(x) \tag{32}
$$

which is a four-vector to give, from (31),

$$
\partial^\mu \mathcal{A}_\mu(x) = 0. \tag{33}
$$

Therefore invariant fermions are found to be

$$\begin{aligned} \psi^{\rm inv}(x) &\equiv \exp\big(ie\int^x A^{\rm L}_\mu(z)dz^\mu\big)\psi(x), \\ \bar\psi^{\rm inv}(x) &\equiv \bar\psi(x)\exp\big(-ie\int^x A^{\rm L}_\mu(z)dz^\mu\big), \end{aligned} \tag{34}$$

which are again path-independent thus local according to (31). In view of (33), the case is called the Lorentz gauge. The Lagrangian reads

$$\mathcal{L} = \bar\psi^{\rm inv}(x)\Big[i\gamma^\mu\Big(\partial_\mu - ie\mathcal{A}_\mu(x)\Big) - m\Big]\psi^{\rm inv}(x) - \frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x), \tag{35}$$

where $F_{\mu\nu}(x) \equiv \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu$ with the constraint (33). This Lagrangian gives rise to the Lorentz force. If use with the mass term, $M^2\mathcal{A}_\mu\mathcal{A}^\mu/2$ can be added. (Note that the massive vector (Proca) field also obeys the constraint (33).) Meanwhile another transformation, in (35),

$$\begin{aligned} \mathcal{A}_\mu(x) &\mapsto \mathcal{A}_\mu(x) + \partial_\mu\theta(x), \\ \Box\,\theta(x) &= 0, \end{aligned} \tag{36}$$

which we shall call as *the null gauge*, prevents the mass term. As for this degree of freedom, there have been some confusions: the $\theta(x)$ is sometime regarded as spurious to be absorbed into a boundary condition of the Green's function [9] but this degree of freedom can also be utilized to prove the gauge invariance of the S-matrix[10]. However it is apparent that $\theta(x)$ does not carry any meaningful degree of freedom if we confine ourselves in the covariant theory: $\theta(x)$ is nothing but the invariant delta function $D(x)$ or $D^{(1)}(x)$. Furthermore in the quantum field theory this transformation is not allowed: the charge operator, $Q^\theta(x^0)$, (which can be obtained by the replacement $\chi(x) \to \theta(x)$ in (3)), cannot annihilate the vacuum $|0\rangle$, $Q^\theta(x^0)|0\rangle \neq 0$, hence, is not well defined.

Owing to the invariant operators (34) as well as $\mathcal{A}_\mu(x)$, we can find the satisfactory LSZ-mapping which states that all Heisenberg operators are expressed in terms of the asymptotic fields satisfying the free field equation,

$$\begin{aligned} &(i\gamma^\mu\partial_\mu - m)\psi^{\rm in}(x) = 0, \\ &\Box\, A^{\rm in}_\mu(x) = 0, \quad \partial^\mu A^{\rm in}_\mu(x) = 0, \end{aligned} \tag{37}$$

such that

$$\begin{aligned} \psi^{\rm inv}(x) &= \psi^{\rm in}(x) + \ldots, \\ \mathcal{A}_\mu(x) &= A^{\rm in}_\mu(x) + \ldots, \end{aligned} \tag{38}$$

where the dots denote the higher order contributions. The both sides of (38) are gauge independent.

As was stressed above, traditionally quantization breaks the gauge symmetry . However our observation allows us to use the standard prescription by Dirac's or by [11] and the Gupta-Bleuler or the Nakanishi-Lautrap formalism for the covariant case [12]; since the conditions, (21), (28), and (33), although fulfilled by the gauge invariant potentials, (19), (26), and (32), respectively, remain unchanged from the conventional gauge conditions.

Finally we make a comment on the functional representation as well as path integral formalism: take the case of $A_0(x) = 0$ gauge in the conventional treatment. There imposes a physical state condition, $\Phi_2(\mathbf{x})|\text{phys}\rangle = 0$, which should be read such that *there is no gauge transformation in the physical space* even in this formalism. The representation cannot be obtained in terms of the usual Hilbert space since $\Phi_2(\mathbf{x})$ is a local operator [13] but can be in the functional (Schrödinger) representation [14],

$$\begin{aligned} \langle\{\phi\}|\hat{\phi}(\mathbf{x}) =& \langle\{\phi\}|\phi(\mathbf{x}), \qquad \langle\{\phi\}|\hat{\pi}(\mathbf{x}) = -i\frac{\delta}{\delta\phi(\mathbf{x})}\langle\{\phi\}|, \\ \hat{\phi}(\mathbf{x}) =& \int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega_{\mathbf{k}}}}\big(a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} + a^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\big), \end{aligned} \tag{39}$$

where a scalar field with a mass m has been introduced for notational simplicty so that $\omega_{\mathbf{k}} \equiv \sqrt{\mathbf{k}^2 + m^2}$ and the caret denotes the operator. However the functional representation consists of infinitely many collections of inequivalent Hilbert spaces; since the inner product to the Fock vacuum, $\langle\{\phi\}|0\rangle \sim \exp\big(-\omega_{\mathbf{k}}\int d^3\mathbf{x}\phi^2(\mathbf{x})/2\big)$, vanishes for arbitrary value of $\phi(\mathbf{x})$, because of the infinite product with respect to $\mathbf{x}$. If we remember that the path integral formula can be obtained with the aid of the functional representation, then it might be easily convinced that *in spite of the fact that in the canonical operator formalism no gauge transformations are allowable, we can move freely from one gauge to others in the path integral* [15].

It would be an interesting task to extend the above idea to the case of non-abelian gauge theories in a similar manner in order to understand the meaning of quark confinement.

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